THE LARGE-SCALE STRUCTURE OF THE UNIVERSE AND QUASI-VORONOI TESSELLATION OF SHOCK FRONTS IN FORCED INVISCID BURGERS' TURBULENCE IN $\mathbf{R}^d$

by

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Abstract. Burgers' turbulence is an accepted formalism for the adhesion model of the large-scale distribution of matter in the Universe. The paper uses variational methods to establish evolution of quasi-Voronoi (curved boundaries) tessellation structure of shock fronts for solutions of the inviscid nonhomogeneous Burgers equation in $\mathbf{R}^d$ in presence of random forcing due to a degenerate potential. The mean rate of growth of the quasi-Voronoi cells is calculated and a scaled limit random tessellation structure is found. Time evolution of the probability that a cell contains a ball of a given radius is also determined.

1. Introduction

It is a well-known, albeit relatively recent, observational fact that matter in the Universe is distributed in cellular "pancake" structures, clusters and superclusters of galaxies, with giant voids between them. The planned wide-ranging Sloan Digital Survey is aimed at providing even finer data about the distribution of galaxies within π steradians (a quarter of the whole sky) to include all the point sources down to the 23rd magnitude and galaxies down to the 19th magnitude ($r' = 18$, where r' is the apparent magnitude in the spectral band with effective wavelength 6280 $\mathring{A}$). It corresponds to about 600 Mpc of the effective depth. Meanwhile, over the last twelve or so years, a major effort was undertaken by the astrophysicists (see the astrophysical literature quoted in references, from Zeldovich, Einasto, Shandarin (1981), through Gurbatov, Malakhov, Saichev (1991), to Bernardeau, Kofman (1995)) provide a mathematical model of an evolution that, starting out with an essentially uniform distribution of matter following the Big Bang, with perhaps minute random quantum fluctuations, would lead to the presently observable rich strucure with filaments, sheets and clusters of galaxies. At this late epoch of the formation of the large scale structure,

- the dark (nonluminous) matter dominates;
- it acts as collisionless dustlike particles;
- no pressure effects need to be taken into account, with the Newtonian gravity being the only force of consequence;
- the radiative and gas dynamics effects are short range.

Assuming the flat, expanding universe, with the scale factor (rate of expansion)

$$a(t) = t^{2/3},$$

and the mean density

$$\bar{\rho} \propto a^{-3},$$

the evolution of the matter density $\rho = \rho(t, \vec{x})$, $\vec{x} \in \mathbf{R}^3$, is usually (see e.g. Peebles (1980), Kofman et al. (1992)) described by the system of three coupled partial differential equations

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla \cdot (\rho \vec{w}) = 0, \tag{1.1}$$

$$\frac{D\vec{w}}{Dt} + H\vec{w} = -\frac{1}{a}\nabla\varphi, \tag{1.2}$$

$$\nabla^2\varphi = 4\pi G a^2(\rho - \bar{\rho}), \tag{1.3}$$

where $\vec{w}$ is the local velocity, φ is the gravitational potential, D/Dt stands for the usual Eulerian derivative, and H and G are, respectively, the Hubble and the gravitational constants. The three equations are, of course, the *continuity equation*, the *Euler equation* and the *Poisson equation.*

This system is not easy to analyze and several attempts have been made at simplifying it, while preserving the predictive ability of the reduced models. Introducing the velocity $\vec{v} = d\vec{x}/da$ in the coordinates co-moving with the expanding Universe (see, e.g., Sahni, Sathyaprakash, Shandarin (1994)), the Euler equation (1.2) is transformed into equation

$$\frac{\partial \vec{v}}{\partial a} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{3}{2a}(\vec{v} + A\nabla\varphi), \tag{1.4}$$

with

$$A = \left(\frac{3}{2}H^2a^3\right)^{-1} = const,$$

where the right-hand side represents, in the Lagrangian approach, the force acting on the particle. It still involves a nonlocal operator so, in 1970, Zeldovich proposed a model where the nonlocal part was simply dropped. This gives a clear Lagrangian picture as (1.4) then becomes the classical Riemann equation, which can explain formation of the pancake structures. This model has been adjusted and studied on the physical level of rigorousness by Gurbatov, Saichev and Shandarin (1984) (see also Shandarin and Zeldovich (1989), Weinberg, Gunn (1990), Bernardeau, Kofman (1995), Kofman, Raga (1992), Kofman et al. (1994)) who replaced the nonlocal term on the right-hand side of (1.4) by the Laplacian, to yield the Burgers equation

$$\frac{\partial \vec{v}}{\partial a} + (\vec{v} \cdot \nabla)\vec{v} = \frac{1}{2}\mu\nabla^2\vec{v}, \qquad \vec{v} \in \mathbf{R}^3, \tag{1.5}$$

where the viscosity term is supposed to mimic the gravitational "adhesion". The constant μ should be small so that the viscosity effects do not affect the motion of matter outside clusters. This *adhesion model* of the large structure of the Universe demonstrates self-organization at large times and has the rough ability to reproduce formation of cellular structures in mass distribution. It has been extensively studied in the astrophysical literature and satisfactorily tested against high resolution (512×512) N-body simulations (Kofman et al. (1994), Melott et al (1994)). In Sahni, Sathyaprakash, Shandarin (1994), extensive simulations and comparisons of different models were conducted. In particular, it made possible evaluation of values of the primordial gravitational potential at the centers of voids that would lead to the currently observable structures. Also, the void sizes were estimated.

One should add that, of course, over the last half-century, because of its ability to reproduce the dynamics of shock formation (see, e.g., Smoller (1994)), the Burgers equation has become one of the standard nonlinear equations of mathematical physics, and found physical interpretations ranging far beyond the turbulence and astrophysical context described above. The applications to nonlinear acoustics, oceanographic, and even traffic flow studies are just some of them and can be found in references provided at the end of this paper. Equation (1.5) is similar to the usual Navier-Stokes hydrodynamic equation, the principal difference being that the pressure term is omitted and that the Burgers equation is usually studied for the class of potential flows which is closed with respect to nonlinear transformation $(\vec{v}\cdot\nabla)\vec{v}$. The similarities and differences between the two equations are discussed in some detail in the introduction to Funaki, Surgailis, Woyczynski (1995).

The present paper provides a rigorous mathematical study of the dynamics of the structure of shocks fronts in the inviscid nonhomogeneous Burgers equation in $\mathbf{R}^d$ in presence of random forcing due to a degenerate potential. The rationale for considering this enhanced model is as follows. The approximation provided by (1.5) near the particle clusters of high density is not that good. To improve it one has to take into account the influence of the clusters through their gravitational field. One simplified way to take these effects into account is to consider the Burgers' equation with the external force field $\vec{F}$:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = \frac{1}{2}\mu\nabla^2\vec{v} - \vec{F}, \qquad \vec{v}\in\mathbf{R}^3. \tag{1.6}$$

The development of the clusters is a relatively slow process and we can assume that the random force $\vec{F}$'s potential Φ is time-independent (and also, of course, stationary in space). For the sake of simplicity it is also assumed to be of the point process type, an assumption reflecting the intermittency of the particle clusters.

As is well known (see Section 2), equation (1.6) can be transformed by the *Hopf-Cole substitution* $\vec{v}(t,\vec{x}) = -\mu\nabla\log u(t,\vec{x})$ into the Schrödinger-type parabolic problem for u:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mu\Delta u + \frac{1}{\mu}\Phi u, \tag{1.7}$$

$$u(0,\vec{x}) = \exp(-\xi/\mu), \qquad \nabla\xi = \vec{v}(0,\vec{x}).$$

To understand the specific influence of the random potential Φ we will consider initially the simplest case, when $\vec{v}(0,\vec{x})\equiv 0,\ \xi = 0,\ u(0,\vec{x})\equiv 1$. As usual, the viscosity μ is a small parameter, i.e., the random potential Φ is large. For fixed t and $\mu\to 0$, it is natural to employ the large deviations theory tools (in the spirit of Schilder (1966) and Freidlin, Wentzel (1984)) and this is the path we are following in this paper.

For $t \to \infty$, one can try a different idea, at least in the one-dimensional case. The operator appearing on the right-hand side of the Schrödinger equation (1.7) has a pure point spectrum $\lambda_1(\omega) > \lambda_2(\omega) > \dots$ in its upper part, with exponentially decaying eigenfunctions $\psi_1(\vec{x}, \omega), \psi_2(\vec{x}, \omega), \dots$, and the eigenvalues $\lambda_i(\omega)$ corresponding to the high peaks of the potential Φ/μ. This suggests the asymptotics

$$u(\vec{x}, t) \approx \sum_i e^{\lambda_i t} \psi_i(\vec{x})(1, \psi_i) \tag{1.8}$$

and

$$\vec{v}(t, \vec{x}) = -\mu \frac{\nabla u(\vec{x}, t)}{u(t, \vec{x})} \approx -\mu \frac{\sum_{i \geq 1} e^{\lambda_i t} \nabla \psi_i(\vec{x})(1, \psi_i)}{\sum_{i \geq 1} e^{\lambda_i t} \psi_i(\vec{x})(1, \psi_i)} \tag{1.9}$$

at least in the vicinity of high peaks, as the corresponding eigenfunctions must be separated in space.

Formulas similar to (1.8-9) require precise information about the structure and distribution of the eigenvalues and eigenfunctions in the upper part of the spectrum corresponding to the high peaks of the potential. Some facts in this area are already known from studies of the intermittency of solutions to the Anderson parabolic problem with random potential (see, e.g., Gärtner, Molchanov (1990)) but a lot remains to be done and we will address these questions in a separate paper.

The problem of evolution of the density fields $\rho(t, \vec{x})$ associated with homogeneous and nonhomogeneous Burgers' equations (1.5) and (1.6) can not be addressed at this point with similar degree of mathematical rigorousness, but can be studied via an approximate model equation (Saichev and Woyczynski (1994 ,1995)) and by a statistical analysis of computer simulations (Janicki, Surgailis and Woyczynski (1995)). It offers a picture complementing the fairly satisfactory understanding of the evolution in the case of unforced Burgers turbulence, at least in dimension one (see Woyczynski (1993), Albeverio, Molchanov, Surgailis (1994), Avellaneda, E (1995), Sinai (1992), Hu, Woyczynski (1994), Surgailis, Woyczynski (1993), Leonenko, Zhanbing (1994), Molchanov, Surgailis, Woyczynski (1995) and Funaki, Surgailis, Woyczynski (1995)). In the case of the forced Burgers equation, there are few mathematically rigorous results, mostly restricted to existence results for a stochastic Burgers equation with additive white noise force (see Bertini, Cancrini and Jona-Lasinio (1994), Holden et al. (1994), and Handa (1995); see also Sinai (1992a)).

The present paper uses variational methods to establish evolution of quasi-Voronoi (curved boundaries) tessellation structure of shock fronts for solutions of the forced Burgers equation (1.6). The preliminary Section 2, describes reduction of the problem to the linear parabolic problem of the Schrödinger type and the related Feynman-Kac formula. Section

3 studies, via the variational Freidlin-Ventzel method, the zero-viscosity limit for a fixed realization of the potential (Theorem 3.1), and the results of this section are made more specific in Theorems 4.1-2, for the degenerate, stick-like potential. The proofs of these theorems are provided in Section 5.

The above results were obtained under assumption of zero initial velocity condition. A brief discussion of what happens for nonhomogeneous initial conditions appears in Section 6.

The last part of the paper is devoted to an analysis of the time-evolution of means of certain geometric parameters of the quasi-Voronoi tessellations discussed in Section 2-6. This is done in the case of random initial singular potential with Poisson locations of the potential peaks. In Section 7, the mean rate of growth of the volume of the quasi-Voronoi cells, or more exactly, the rate of decay of the number of cells per unit volume is calculated (Theorem 7.1) and, surprisingly, we also discover a scaling limit of the random quasi-Voronoi tessellation structure itself (Theorem 7.2). The time-evolution of the probability that a cell contains a ball of a given radius, another interesting parameter of the tessellation structure, is also determined (Theorem 7.3).

We should also mention that the nonstandard Voronoi tessellations have already appeared in the literature in connection with the study of crystal growth in the case when crystal seeds appear at different times or grow with different speeds (see, e.g., Møller (1994)). Also, in the astrophysical context, the Leyden Ph.D. thesis of Van de Weygaert (1991) provided a computer based analysis of intergalactic voids in terms of their tessellation structure. None of them, however, dealt with a rigorous analysis of a complex random dynamical system described in terms of the underlying stochastic nonlinear partial differential equations, the main objective of the present paper.

2. Preliminaries

Consider the Cauchy problem for the Burgers' equation with forcing

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} = \frac{1}{2}\mu\Delta\vec{v} - \nabla\Phi, \tag{2.1}$$

$$\vec{v}(0,x) \equiv -\nabla S_0(x),$$

for the velocity field $\vec{v} = \vec{v}(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}^d$, where $\Phi = \Phi(x)$ and $S_0(x)$ are given potential fields (to lessen the clutter in what follows, we will suppress arrows over the position vector x). Under certain smoothness and boundedness conditions on the potentials Φ and S_0 (for example, like those specified in Theorem 3.1), the Hopf-Cole substitution

$$\vec{v}(t,x) = -\mu\nabla \log u(t,x), \tag{2.2}$$

reduces (2.1) to the Cauchy problem for a linear parabolic equation of the Schrödinger type

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mu\Delta u + \frac{1}{\mu}\Phi u, \tag{2.3}$$
$$u(0,x) \equiv e^{S_0(x)/\mu}.$$

Its solution is given by the usual *Feynman-Kac formula*

$$u(t,x) = E^W\left[\exp\left(\frac{1}{\mu}\int_0^t \Phi(x+\sqrt{\mu}W_s)ds + \frac{1}{\mu}S_0(x+\sqrt{\mu}W_t)\right)\right], \tag{2.4}$$

where $E^W[\ldots]$ is the expectation over trajectories of the standard Wiener process $W_s, s \geq 0, W_0 = 0$ in $\mathbf{R}^d$ (see, e.g., Freidlin, Wentzel (1984), Carmona and Lacroix (1990)). Substituting (2.4) into (2.2), one obtains

$$\vec{v}(t,x) = -\left(E^W\left[\exp\left(\frac{1}{\mu}\int_0^t \Phi(x+\sqrt{\mu}W_s)ds + \frac{1}{\mu}S_0(x+\sqrt{\mu}W_t)\right)\right]\right)^{-1} \times \tag{2.5}$$
$$\times E^W\left[\int_0^t \left(\nabla\Phi(x+\sqrt{\mu}W_s)\, ds + \nabla S_0(x+\sqrt{\mu}W_t)\right) \times\right.$$
$$\left.\times \exp\left(\frac{1}{\mu}\int_0^t \Phi(x+\sqrt{\mu}W_s)ds + \frac{1}{\mu}S_0(x+\sqrt{\mu}W_t)\right)\right].$$

3. Zero Viscosity Limit

For $\mu = 0$, equation (2.1) reduces to the *Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} - \frac{1}{2}(\nabla S, \nabla S) = \Phi, \tag{3.1}$$
$$S(0,x) = S_0(x),$$

for the velocity potential $S(t,x),\ (t,x) \in [0,\infty) \times \mathbf{R}^d$ satisfying

$$\vec{v}(t,x) = -\nabla S(t,x), \qquad \vec{v}(0,x) = -\nabla S_0(x). \tag{3.2}$$

The solution of (3.1) is given by

$$S(t,x) = \sup_{\gamma\in\Gamma_{x,t}} S(t,x;\gamma), \tag{3.3}$$

where the action functional

$$S(t,x;\gamma)=\int_0^t\Big(\Phi(\gamma(s))-\frac{1}{2}|\dot\gamma(s)|^2\Big)ds+S_0(\gamma(t)) \tag{3.4}$$

is the difference of the potential and kinetic energy, and the supremum is taken in the class $\Gamma_{x,t}$ of all paths

$$\gamma:[0,t]\longmapsto\mathbf{R}^d,\qquad\gamma(0)=x,$$

which are absolutely continuous and satisfying the condition

$$\int_0^t|\dot\gamma(s)|^2ds<\infty. \qquad (3.5).$$

In particular, for $\Phi\equiv 0$, the extremal (Lagrangian) paths are linear:

$$\gamma(s)=x+\frac{(y-x)s}{t}$$

yielding the well known "geometric" solution

$$S(t,x)=\sup_y\Big(S_0(y)-\frac{1}{2t}|x-y|^2\Big). \tag{3.6}$$

However, the physical inviscid (limit) solution of equation (2.1) is defined as a limit of the Hopf-Cole solution (2.5) for $\mu\to 0$ (see, e.g., Vergassola, Dubrulle, Frisch, Nullez (1994)). Clearly, finding the limit is related to the variational problem of maximizing the integral in the exponent of formula (2.5), and it is this problem that we will concentrate on in what follows. In the case of the homogeneous Burgers equation with random initial condition, an analogous, but much simpler extremal problem was discussed in Albeverio, Molchanov, Surgailis (1994) and Molchanov, Surgailis, Woyczynski (1995).

Consider the variational problem (3.3-4) and assume that $\Phi(.),S_0(.)\in C^1(\mathbf{R}^d)$ and that the extremal path $\gamma^*\in\Gamma_{x,t}$ in (3.3) exists and is unique. Then, as is well known (see Courant, Hilbert (1953)) the extremal path γ^* satisfies the Euler equation

$$\ddot\gamma^*(s)=-\nabla\Phi(\gamma^*(s)),\qquad s\in[0,t], \tag{3.7}$$

and the boundary conditions

$$\gamma^*(0)=x,\qquad\dot\gamma^*(t)=\nabla S_0(\gamma^*(t)). \tag{3.8}$$

Note that, for $S_0\equiv 0$, the extremal path *stops* at the end of time $s=t$, or perhaps earlier.

Theorem 3.1. *Let* $\Phi(.), S_0(.) \in C^1(\mathbf{R}^d)$,

$$\Phi(x) \le C + C_1|x|^2, \tag{3.9}$$

$$S_0(x) \le C + C_2|x|^2, \tag{3.10}$$

and

$$|\nabla\Phi(x)| \le Ce^{C|x|^2}, \tag{3.11}$$

where $C, C_1, C_2 < \infty$ *are constants with* $C_1 < (4t^2)^{-1}, C_2 < (4t)^{-1}$*. Furthermore, suppose that the variational problem (3.3) has a unique solution* $\gamma^* \in \Gamma_{x,t}$*. Then, the zero viscosity limit solution*

$$\vec{v}(t,x) = \lim_{\mu\to 0} \vec{v}(t,x;\mu) \tag{3.12}$$

of Burgers' equation (2.1) exists, and is given by

$$\vec{v}(t,x) = -\int_0^t \nabla\Phi(\gamma^*(s))\,ds - \nabla S_0(\gamma^*(t)) = -\dot{\gamma}^*(0). \tag{3.13}$$

Proof. Set

$$T_\mu(t,x) = \mu^{-1}\left(\int_0^t \Phi(x+\sqrt{\mu}W_s)\,ds + S_0(x+\sqrt{\mu}W_t)\right)$$

where $W_s, s \ge 0,\ W_0 = 0$, is the Wiener process (see (2.4)). Let $\Psi(x), \Psi_0(x),\ x \in \mathbf{R}^d$, be continuous, possible vector-valued, functions such that

$$|\Psi(x)| + |\Psi_0(x)| \le Ce^{C|x|^2},$$

for some constant $C < \infty$. Then

$$\lim_{\mu\to 0}\frac{E^W[(\int_0^t \Psi(x+\sqrt{\mu}W_s)\,ds + \Psi_0(x+\sqrt{\mu}W_t))e^{T_\mu(t,x)}]}{E^W[e^{T_\mu(t,x)}]} = \int_0^t \Psi(\gamma^*(s))\,ds + \Psi_0(\gamma^*(t)). \tag{3.14}$$

Relation (3.14) can be proved as in Schilder (1965), Theorem A, where only the one-dimensional case $d = 1$ was considered; see also Freidlin, Wentzell (1984). From (3.14) and the Feynman-Kac formula (2.5), with $\Psi(x) = \nabla\Phi(x)$, $\Psi_0(x) = \nabla S_0(x)$, one immediately obtains the existence of the limit (3.12) and the first equality of (3.13). Furthermore, from (3.7) and the boundary conditions (3.8), it follows that

$$\int_0^t \nabla\Phi(\gamma^*(s))\,ds + \nabla S_0(\gamma^*(t)) = -\int_0^t \ddot{\gamma}^*(s)\,ds + \nabla S_0(\gamma^*(t)) = \dot{\gamma}^*(0).$$

Remark 3.1. If, in addition to the conditions of Theorem 3.1., one assumes that the variational problem (3.3) has a unique solution for every x from an open neighborhood U then $S(t,\,.\,) \in C^1(U)$ and

$$-\nabla S(t,x) = \vec{v}(t,x),$$

where $\vec{v}(t,x)$ is given by (3.13). Also, it is worthwhile to note that the results of this section can be properly interpreted within the framework of viscosity solutions for general Hamilton-Jacobi equations (see, e.g., Lions (1982), Chapter 11) which, however, we deemed unnecessary in our relatively simple situation of the Burgers equation.

4. The case of point potential

In the present section our aim is to obtain an explicit description, including the structure of the shock-fronts (discontinuities), of the zero-viscosity solution (3.13) in the case of the degenerate "discrete" potential

$$\Phi(x) = \sum_{j\in I} h_j \mathbf{1}(x = x_j), \qquad x \in \mathbf{R}^d, \tag{4.1}$$

which is a superposition of zero-volume "sticks" of height $h_j > 0$ located at points x_j. The index set I is assumed to be countable and the set set $\{x_j\}_{j\in I} \subset \mathbf{R}^d$, is assumed locally finite. To simplify the problem, we consider the case of zero initial velocity, or $S_0(x) = 0$, although a discrete potential $S_0(x)$ of a similar form can easily be included (see Section 8). The explicit form of our solutions also permits us to study the evolution of their discontinuities ('shock fronts').

Obviously, Theorem 3.1 can not be applied directly, nor can equation (3.13), since (4.1) is not even continuous. Thus, a natural approach is to approximate $\Phi(x)$ appearing in formula (4.1), by smooth potentials $\Phi_n(x)$ converging to $\Phi(x)$ in a certain sense, and then to define the inviscid solution $\vec{v}(t,x)$ as the limit of corresponding solutions $\vec{v}_n(t,x)$, i.e.

$$\vec{v}(t,x) = \lim_{n\to\infty} \vec{v}_n(t,x), \tag{4.2}$$

where

$$\vec{v}_n(t,x) = -\int_0^t \nabla\Phi_n(\gamma_n^*(s))\,ds = -\dot{\gamma}_n^*(0), \tag{4.3}$$

and where $\gamma_n^* \in \Gamma_{x,t}$ is the solution of the variational problem (3.3), with $\Phi(.)$ replaced by $\Phi_n(.)$, and $S_0(x) \equiv 0$.

Theorem 4.1. *For $\Phi(x)$ from (4.1) and satisfying condition (3.9), the maximal action functional*

$$S(t,x) \equiv \sup_{\gamma\in\Gamma_{x,t}} \int_0^t \left(\sum_j h_j \mathbf{1}(\gamma_s = x) - \frac{1}{2}|\dot\gamma(s)|^2\right) ds$$

$$= \sup_j \Big(th_j - \sqrt{2h_j}|x-x_j|\Big) \vee 0 \tag{4.4}$$

is the upper envelope (i.e., supremum) of cones

$$c_j(t,x) = \Big(th_j - \sqrt{2h_j}|x-x_j|\Big) \vee 0 \tag{4.4a}$$

of height th_j and centered at x_j.

Theorem 4.2. *Assume that the following four conditions hold true:*

(i) Approximating potentials $\Phi_n(.) \in C^1(\mathbf{R}^d)$, $n \geq 1$, and they satisfy conditions (3.9) and (3.11) of Theorem 3.1;

(ii) There exist unique solutions γ_n^*, and $\gamma^* \in \Gamma_{x,t}$ of the variational problem (3.3) corresponding to potentials $\Phi_n(.)$ and $\Phi(.)$; respectively.

(iii) Potentials $\Phi_n(x) \to \Phi(x)$ decrease monotonically for each $x \in \mathbf{R}^d$;

(iv) Gradients $\nabla\Phi_n(x) \to 0$ uniformly on each compact set in $\mathbf{R}^d \setminus \{x_j\}_{j\in I}$.

Then, for a given $t > 0$ and $x \in \mathbf{R}^d$, $x \notin \{x_j\}_{j\in I}$, the limit relation (4.2) is valid, and the limit solution

$$\vec{v}(t,x) = \begin{cases} \sqrt{2h_{j^*}}\frac{x_{j^*}-x}{|x_{j^*}-x|}, & \text{if } h_{j^*} > \frac{2|x_{j^*}-x|^2}{t^2}; \\ 0, & \text{otherwise}, \end{cases} \tag{4.5}$$

where (x_{j^*}, h_{j^*}) is the point which maximizes the corresponding action (4.4), i.e.,

$$(th_{j^*} - \sqrt{2h_{j^*}}|x-x_{j^*}|) = \sup_j (th_j - \sqrt{2h_j}|x-x_j|). \tag{4.6}$$

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Fig.1. Shock fronts form a quasi-Voronoi tessellation. The boundaries between black and white areas are level curves for the action cones from Theorem 4.1. From Janicki, Surgailis, Woyczynski (1995).

Remark 4.1. Note, that limit velocity (4.5) equals

$$\vec{v}(t,x) = -\nabla S(t,x), \tag{4.7}$$

with $S(t,x)$ given by (4.4). In particular, the discontinuities of (4.7) correspond to intersections of the cones $c_j(t,x)$ with other cones or with the zero level $\Phi(x) = 0$. If all heights $h_1 = h_2 = \ldots$ are equal and the (bases of) cones cover all space $\mathbf{R}^d$, the set of discontinuities is independent of t and coincides with the classical straight-edged *Voronoi tessellation* of $\mathbf{R}^d$ with centers x_j. Different heights $h_i \neq h_j$, lead to a more complicated *quasi-Voronoi tessellation* with curved boundaries (see Fig. 1). For general information on the subject of Voronoi tessellations we refer to the monograph by Okabe, Boots, Sugihara (1992) and the lecture notes by Møller (1994).

In its general features, this picture agrees with the situation observed in the unforced Burgers turbulence at large Reynolds numbers (see e.g. Kraichnan (1959, 1968), Gurbatov, Malakhov, Saichev (1991), Albeverio, Molchanov, Surgailis (1994), Molchanov, Surgailis, Woyczynski (1995), etc.) with the important difference that in the case of forced turbulence, the velocity *does not* decay in time—it remains constant as long as x belongs to the same Voronoi cell. At the moment in the time-evolution of the system when the cell is 'engulfed' by a larger one, the velocity increases in absolute value by an amount proportional to the square root of the height of the cone of the larger cell at time $t = 1$.

The important question of the existence of stationary solutions in the forced Burgers turbulence, that is the situation when forcing and dissipation eventually balance each other in the statistical sense, is not discussed in this paper (see, however, Saichev, Woyczynski (1995) and Sinai (1992a), and the references therein).

5. Proofs of Theorems 4.1-2

Proof of Theorem 4.1. Initially, consider the case when the set $\{x_j\}_{j\in I}$ consists of a single point x_1, i.e.

$$\Phi(x) = h_1 \mathbf{1}(x = x_1). \tag{5.1}$$

We want to show that

$$S(t,x) = \begin{cases} th_1 - \sqrt{2h_1}|x - x_1|, & \text{if } th_1 > \sqrt{2h_1}|x - x_1|; \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

It is easy to check that the right-hand side of (5.2) equals the action along the linear motion from x to x_1 with constant speed $|v| = \sqrt{2h_1}$, until reaching x_1, and then staying at x_1 for the rest of time, or the action for the trivial trajectory $\gamma \equiv x$, depending on which of the two cases take place.

To prove (5.2), assume that $\gamma \in \Gamma_{x,t}$ does not visit x_1. Then,

$$S(t,x;\gamma) = h_1 \int_0^t \mathbf{1}(\gamma(s) = x_1)\, ds - \frac{1}{2}\int_0^t |\dot\gamma(s)|^2 ds < 0,$$

unless $\gamma(s) \equiv x$. Hence, the optimal trajectory $\gamma^* \in \Gamma_{x,t}$, whenever it exists, either stays at x all the time, or visits x_1. In the latter case, γ^* obviously remains at x_1 after first hitting it. In other words, if $\gamma^*(s) \not\equiv x$, then set $\tau_1 = \inf\{s : \gamma^*(s) = x_1\} \le t$, and

$$\gamma^*(s) = x_1, \qquad \tau_1 \le s \le t. \tag{5.3}$$

By Cauchy-Schwartz' inequality, for any $\gamma \in \Gamma_{x,t}$,

$$\begin{aligned} \int_0^{\tau_1} |\dot\gamma(s)|^2 ds &\ge \tau_1^{-1} \left| \int_0^{\tau_1} \dot\gamma(s) ds \right|^2 \\ &= \tau_1^{-1} |\gamma(\tau_1) - \gamma(0)|^2 \\ &= \tau_1^{-1} |x_1 - x|^2 = \tau_1 |v|^2, \end{aligned} \tag{5.4}$$

where

$$v = \frac{x_1 - x}{\tau_1}. \tag{5.5}$$

Hence, the trajectory that minimizes the left-hand side of (5.4), has to move from x to x_1 with constant velocity (5.5). To find τ_1, let us maximize the corresponding action

$$\begin{aligned} S(t,x;\gamma) &= h_1(\tau - \tau_1) - \frac{1}{2}|v|^2\tau_1 \\ &= h_1(\tau - \tau_1) - \frac{1}{2}\tau_1^{-1}|x - x_1|^2 \end{aligned} \tag{5.6}$$

over such rectilinear paths. This yields

$$\tau_1 = \frac{|x - x_1|}{\sqrt{2h_1}}, \tag{5.7}$$

or

$$|v| = \sqrt{2h_1}. \tag{5.8}$$

This proves the special case (5.2).

Now, consider the general case of potential $\Phi(x)$ defined in (4.1). It is clear from the above discussion that $S(t,x)$ is not smaller than the right-hand side of (4.4). Moreover, the latter is finite and the supremum, unless zero, is achieved for some cone $c_{j^*}(t,x)$, which follows from condition (3.9) and the fact that $\{x_j\}$ is locally finite. Indeed, $c_j(t,x) = 0$ unless $\Phi(x_j) = h_j > 2|x - x_j|^2/t^2$ or, according to (3.9), unless $C + |x_j|^2/(4t^2) > 2|x - x_j|^2/t^2$. The last inequality implies $|x_j| < C_4$ for some $C_4 = C_4(t,x,C) < \infty$ and any t, x, C fixed, i.e., $c_j(t,x) = 0$ for all but finitely many points x_j in view of the assumption of local finiteness of $\{x_j\}_{j\in I}$. Hence, we can assume, without loss of generality that the set $\{x_j\}_{j\in I} \equiv \{x_j\}$ is finite. Indeed, adding new points to the set $\{x_j\}$ can only increase the left-hand side of (4.4), while the right-hand side remains the same ($= c_{j^*}(t,x)$).

For any subset $\{y_j\} \subset \{x_j\}$, and any $0 \le \sigma \le t$, introduce the class $\Gamma_{x,t}(\{y_j\},\sigma) \subset \Gamma_{x,t}$ of all paths which visit *all* points of $\{y_j\}$ and which stay at those points total time σ. In other words,

$$\Gamma_{x,t}(\{y_j\}) = \Big\{\gamma \in \Gamma_{x,t} :\ \exists\, \tau_j \in [0,t] \text{ s.t. } \gamma(\tau_j) = y_j\ \forall j \text{ and } \sum_j \int_0^t \mathbf{1}[\gamma(s) = y_j]\, ds = \sigma\Big\}. \tag{5.9}$$

Let $y_{j^*} \in \{y_j\}$ be the point of the maximal peak in this set (we suppose, for simplicity, that it is unique), i.e.,

$$h_{j^*} = \max\{h_j :\ y_j \in \{y_j\}\}. \tag{5.10}$$

Then, if $\gamma^* \in \Gamma_{x,t}$ is the optimal trajectory and $\gamma^* \in \Gamma_{x,t}(\{y_j\}, \sigma)$ for some subset $\{y_j\} \subset \{x_j\}$, then γ^* has to stay at y_{j^*} after it first hits it, as otherwise the action will decrease:

$$h_{j^*}(t - \tau_{j^*}) > \int_{\tau_{j^*}}^{t} \left(\sum_j h_j \mathbf{1}[\gamma(s) = y_j] - \frac{1}{2}|\dot{\gamma}(s)|^2 \right) ds,$$

for any

$$\gamma \in \Gamma_{x,t}, \quad \gamma(\tau_{j^*}) = y_{j^*}, \quad \gamma(s) \not\equiv y_{j^*}, \; s \in [\tau_{j^*}, t].$$

Let

$$\Gamma^*_{x,t}(\{y_j\}, \sigma) \subset \Gamma_{x,t}(\{y_j\}, \sigma)$$

be the set of trajectories $\gamma(s)$ having the property that γ visits y_{j^*} as its last point, and then stays at it until time t.

By the above argument, one can find $\{y_j\} \subset \{x_j\}$ and $0 \le \sigma \le t$ such that

$$S(t,x) = \sup_{\gamma \in \Gamma^*_{x,t}(\{y_j\}, \sigma)} S(t,x;\gamma). \tag{5.11}$$

Let $\{y_j\} = \{y_1, \ldots, y_n\}$ so that $y_{j^*} = y_n$. Note that, for any $\gamma \in \Gamma^*_{x,t}(\{y_j\}, \sigma)$,

$$S(t,x;\gamma) \le S(t,x;\tilde{\gamma}), \tag{5.12}$$

where $\tilde{\gamma} \in \Gamma^*_{x,t}(\{y_j\}, \sigma)$ visits the points $y_1, \ldots, y_n$ in the same order as γ, but stays the *whole* time σ at $y_{j^*} = y_n$. Note, that $\tilde{\gamma}$ can be easily constructed by pasting together the parts of γ between the visits, and putting $\tilde{\gamma}(s) = y_n$ for $s \in [t-\sigma, t]$. This, however, leads to the situation discussed at the beginning of the proof, where the potential had a single peak at y_n: instead of going along $\tilde{\gamma}$, it makes more sense to go straight to y_n with speed $|v| = \sqrt{2h_n}$, and stay there afterwards. The corresponding action is then given by (5.2), with h_1, x_1 replaced by h_n, y_n, respectively, and y_n chosen among all x_j's so that the action is minimal. This proves (4.4), and Theorem 4.1.

Proof of Theorem 4.2. Clearly, the right-hand side of (4.5) coincides with $\dot{\gamma}^*(0)$; see the above proof of Theorem 4.1. Therefore, by (3.13), the convergence (4.2) is equivalent to

$$\dot{\gamma}^*(0) = \lim_{n\to\infty} \dot{\gamma}^*_n(0) \tag{5.13}$$

Let us show that the sequence $\{\gamma^*_n(.)\}$ is relatively compact in $C([0,t];\mathbf{R}^d)$, which follows from the condition

$$\sup_n \int_0^t |\dot{\gamma}^*_n(s)|^2 ds < \infty; \tag{5.14}$$

see Freidlin, Wentzell (1984), p. 78. Write $S_n(t,x), S_n(t,x;\gamma)$ for the action functionals (3.3-4), with $\Phi(.)$ replaced by $\Phi_n(.)$, and $S_0(.) \equiv 0$. Since $S_n(t,x) = S_n(t,x;\gamma_n^*) \geq S_n(t,x;\gamma(.) \equiv x) = \Phi_n(x) \geq \Phi(x) \geq 0$,

$$\begin{aligned}
\frac{1}{2}\int_0^t |\dot\gamma_n^*(s)|^2 ds &= -S_n(t,x) + \int_0^t \Phi_n(\gamma_n^*(s))\, ds \\
&\leq \int_0^t \Phi_n(\gamma_n^*(s))\, ds \\
&\leq \int_0^t (C + C_1|\gamma_n^*(s)|^2)\, ds \qquad \text{(see (3.9), Theorem 4.2}(i)) \\
&= \int_0^t (C + C_1\Big|x + \int_0^s \dot\gamma_n^*(u)\, du\Big|^2)\, ds \qquad (5.15) \\
&\leq Ct + 2C_1|x|^2 t + 2C_1 \int_0^t \Big|\int_0^s \dot\gamma_n^* du\Big|^2 ds \\
&\leq C_3 + 2C_1 \int_0^t s \int_0^s |\dot\gamma_n^*|^2 du\, ds \\
&\leq C_3 + C_1 t^2 \int_0^t |\dot\gamma_n^*(u)|^2 du,
\end{aligned}$$

where $C_3 = C_3(t,x) < \infty$ and C_1 are independent of n; see Theorem 4.2, *(i)*. As $C_1 < (4t^2)^{-1}$, this proves (5.14). Consequently, without loss of generality, we can assume that there exists $\gamma_\infty^* \in \Gamma_{x,t}$ such that

$$\gamma_n^* \longrightarrow \gamma_\infty^* \qquad \text{in } C([0,t];\mathbf{R}^d). \tag{5.16}$$

We claim that

$$\gamma_\infty^* = \gamma^*, \tag{5.17}$$

which follows from the fact that

$$S(t,x;\gamma_\infty^*) = S(t,x), \tag{5.18}$$

and the assumption *(ii)* in Theorem 4.2, to the effect that the least action is achieved at a unique point. In turn, (5.18) follows from the inequalities

$$\limsup_{n\to\infty} S_n(t,x;\gamma_n^*) \leq S(t,x;\gamma_\infty^*) \tag{5.19}$$

and

$$S(t,x) \leq S_n(t,x;\gamma_n^*), \tag{5.20}$$

which will be proved below.

Let us prove inequality (5.19) first. By (5.16),

$$\int_0^t |\dot\gamma^*_\infty(s)|^2 ds \le \liminf_{n\to\infty} \int_0^t |\dot\gamma^*_n(s)|^2 ds, \tag{5.21}$$

see Freidlin, Wentzell (1989), Lemma 2.1(a). Next, as $\Phi_n(x)$ are continuous and monotonically decrease to $\Phi(x)$, it is easy to show that for any $K < \infty$, any $\epsilon > 0$ and $\delta > 0$, one can find an n_0 such that for all $n > n_0$,

$$\Phi_n(x) \le \begin{cases} \epsilon, & \text{if } |x - x_j| \ge \delta,\ |x| \le K; \\ h_j + \epsilon, & \text{if } |x - x_j| \le \delta,\ |x| \le K. \end{cases} \tag{5.22}$$

Therefore, and with (5.16) in mind,

$$\begin{aligned} \int_0^t \Phi_n(\gamma^*_n(s))\, ds &\le \epsilon t + \sum_j (h_j + \epsilon) \int_0^t \mathbf{1}[|\gamma^*_n(s) - x_j| < \delta]\, ds \\ &\le \epsilon t + \sum_j (h_j + \epsilon) \int_0^t \mathbf{1}[|\gamma^*_\infty(s) - x_j| < 2\delta]\, ds, \end{aligned} \tag{5.23}$$

provided $n > n_0$ is chosen sufficiently large. As $\epsilon > 0$ and $\delta > 0$ are arbitrary, from (5.21) and (5.23) we infer (5.19).

On the other hand, (5.20) follows easily from the inequality $\Phi_n(x) \ge \Phi(x)$, which holds true for all $x \in \mathbf{R}^d$, and which implies, of course, that

$$S_n(t, x; \gamma) \ge S(t, x; \gamma)$$

for any $\gamma \in \Gamma_{x,t}$, and consequently

$$S_n(t, x) = S_n(t, x; \gamma^*_n) \ge S(t, x; \gamma)$$

for any $\gamma \in \Gamma_{x,t}$, including $\gamma = \gamma^*$, which yields (5.20).

It remains to prove (5.13). By (5.16-17),

$$\gamma^*_n \longrightarrow \gamma^* \qquad \text{in } C([0, t]; \mathbf{R}^d), \tag{5.24}$$

where $\gamma^*(s)$ is a rectilinear motion from x, with a constant velocity $v = \dot\gamma^*(0)$, at least for some time interval $0 \le s \le \tau \le t$. Write

$$\Big(\gamma^*_n(s) - \gamma^*_n(0)\Big) - \Big(\gamma^*(s) - \gamma^*(0)\Big) = \int_0^s \Big(\dot\gamma^*_n(u) - v\Big)\, du \tag{5.25}$$

$$= \int_0^s \left(\dot{\gamma}_n^*(u) - \dot{\gamma}_n^*(0)\right) du + s((\dot{\gamma}_n^*(0) - v).$$

For fixed $t \geq s > 0$, the left-hand side of (4.25) tends to 0 as $n \to \infty$ according to (5.24). Hence, it remains to show that, for some $s > 0$ (s can be arbitrarily small),

$$\lim_{n\to\infty} \int_0^s \left(\dot{\gamma}_n^*(u) - \dot{\gamma}_n^*(0)\right) du = 0, \tag{5.26}$$

or that

$$\sup_{0\leq u\leq s} |\dot{\gamma}_n^*(u) - \dot{\gamma}_n^*(0)| \longrightarrow 0 \qquad (n \to \infty). \tag{5.27}$$

But

$$\begin{aligned} |\dot{\gamma}_n^*(u) - \dot{\gamma}_n^*(0)| &= \left| \int_0^u \ddot{\gamma}_n^*(r)\, dr \right| \\ &= \left| \int_0^u \nabla\Phi_n(\gamma_n^*(r))\, dr \right| \\ &\leq u \cdot \sup_{|x-y|\leq\delta} |\nabla\Phi_n(y)|, \end{aligned} \tag{5.28}$$

where $\delta > 0$ is chosen so that

$$\sup_{0\leq r\leq s} |\gamma_n^*(r) - x| \leq \delta. \tag{5.29}$$

Let $x \notin \{x_j\}$. Then, using (5.24) for a given $\delta < (1/2)\,\mathrm{dist}\,(x, \{x_j\})$, one can find $s > 0$ and $n_0 > 0$ such that (5.29) holds for all $n > n_0$. According to assumption (iv) of Theorem 4.2,

$$\sup_{|x-y|\leq\delta} |\nabla\Phi_n(y)| \longrightarrow 0 \qquad (n \to \infty),$$

which proves (5.27) in view of (5.28). Thus, the proof of Theorem 4.2 is complete.

6. Random Poisson potential: evolution of the cell structure.

Below, we consider the case of a random Poisson point potential $\Phi(x)$ (see (4.1)), i.e., we assume that $\{(x_j, h_j)\}$ is a (marked) Poisson process in $\mathbf{R}^{d+1}$, with intensity measure $\lambda\, dx\, dF(h)$, with F being a probability distribution function on $\mathbf{R}_+ = (0, \infty)$, and $\lambda > 0$ being a parameter. In other words, $\{x_j\}$ is a homogeneous Poisson process in $\mathbf{R}^d$ with intensity λ, while $h_j > 0$ are independent and identically distributed according to F. We assume that F satisfies condition

$$\int_{\mathbf{R}^d} \left(1 - F(|x|^2)\right) dx < \infty, \tag{6.1}$$

which guarantees the growth condition (3.9), or finiteness of the action functional $S(t,x)$, for almost all realizations $\{(x_j,h_j)\}$. Indeed, for given constants $K, K_1 < \infty$, consider the set

$$A_{K,K_1} := \{\{(x_j,h_j)\} : h_j < K + K_1|x_j|^2 \text{ for all } j\}.$$

Then, by the well-known properties of the Poisson process,

$$\begin{aligned} P(A_{K,K_1}) &= E\prod_j F(K+K_1|x_j|^2) \\ &= E\exp\Big[\sum_j \log F(K+K_1|x_j|^2)\Big] \\ &= \exp\left[-\lambda K_1^{-d/2}\int_{\mathbf{R}^d}(1-F(K+|x|^2))\,dx\right], \end{aligned} \tag{6.2}$$

which, in view of (6.1), implies that $P(A_{K,K_1}) \to 1$ as $K\to\infty$, so that, consequently,

$$P(\lim_{K\to\infty} A_{K,K_1}) = \lim_{K\to\infty} P(A_{K,K_1}) = 1.$$

Note, that in the final expression of (6.2) the negative of the term inside the exponential represents the total intensity of "bad" events $\{\{(x_j,h_j)\} : h_j \ge K + K_1|x_j|^2\}$.

With each point (x_j,h_j) we associate a, possibly empty, set

$$C_j(t) = \{x \in \mathbf{R}^d : c_j(t,x) = S(t,x)\}, \tag{6.3}$$

where $c_j(t,x)$ is the corresponding cone (4.4a) with the vertex at (x_j,h_j). Any connected component $C \subset C_j(t)$ will be called a *cell.* We shall distinguish between the *first order cells*, containing the base x_j of the vertex as an interior point, and the rest, which we call *second order cells.* Apparently, only first order cells are important in the formation of the cellular structure in the presence of forcing due to a degenerate potential (4.1).

Let $\{(x_j^*(t),h_j^*(t))\}$ be the point process of vertices of *first order cones*, i.e., the cones above first order cells. In other words, for any $f \in C_0(\mathbf{R}^d\times\mathbf{R})$,

$$\sum_j f(x_j^*(t),h_j^*(t)) = \sum_j f(x_j,h_j)\mathbf{1}(th_j > th_k - \sqrt{2h_k}|x_j-x_k|, \text{ for all } k\ne j). \tag{6.4}$$

The introduced process is a subprocess of the original Poisson process $\{(x_j,h_j)\}$; it is strictly stationary but *not* Poisson (unless $t=0$). By the ergodicity of the Poisson process, for any fixed $t>0$, the process $\{(x_j^*(t),h_j^*(t))\}$ is also ergodic (the σ-algebra of its shift-invariant sets is contained in the corresponding σ-algebra of the Poisson process, see also Surgailis (1981), Remark 3.2), with intensity

$$\lambda(t) = E\,N((0,1]^d;t) = \lim_{R\to\infty}(2R)^{-d}N((-R,R]^d;t), \tag{6.5}$$

where $N(A;t) = \#\{j : x_j^*(t) \in A\}$ is the number of points in $A \subset \mathbf{R}^d$.

To analytically evaluate $\lambda(t)$, we shall introduce random variables $g_j = \sqrt{h_j}$, with the distribution function

$$P(g_j \le u) = G(u) = F(u^2). \tag{6.6}$$

Using the relation $\lambda(t) = P(N(dx;t) = 1)/dx$, we obtain

$$\begin{aligned}\lambda(t) &= P\big(x_j \in dx, tg_j^2 > tg_k^2 - \sqrt{2}g_k|x_j - x_k|, \text{ for all } k \ne j\big)/dx \\ &= \lambda P\big(g_k < |x_k - x|/\sqrt{2}t + \sqrt{|x_k - x|^2/2t^2 + g_j^2}, \text{ for all } k \ne j \;\big|\; x_j \in dx\big) \\ &= \lambda \int_0^\infty E \prod_k G\big(|x_k - x|/\sqrt{2}t + \sqrt{|x_k - x|^2/2t^2 + u^2}\big)\, dG(u),\end{aligned}$$

where we can put $x = 0$. Hence, as in (6.2),

$$\lambda(t) = \lambda \int_0^\infty \exp\Big[-\lambda(\sqrt{2}t)^d \int_{\mathbf{R}^d} (1 - G)\big(|x| + \sqrt{|x|^2 + u^2}\big)\, dx\Big]\, dG(u). \tag{6.7}$$

The last formula implies that $\lambda(t) \to 0$ as $t \to \infty$; moreover, the decay rate is determined by tail behavior of the probability distribution function $G(u)$ as $u \to \infty$ (the probabilities of "high peaks"). The decay rate of $\lambda(t)$ can be rigorously obtained under the assumption that $G(.)$ is asymptotically *max-stable* (see below, also Albeverio, Molchanov, Surgailis (1994), Section 5). The inverse $1/\lambda^{1/d}(t)$ gives the order of the typical distance between the first-order cones, or the *linear scale of cells* of the quasi-Voronoi tessellation. (Indeed, we do know that there are on the average $\lambda(t) \cdot \mathrm{Leb}\,(A)$ first-order cones in a large box $A \subset \mathbf{R}^d$ so, assuming that they are positioned more or less regularly, the typical distance between them should be of the order $1/\lambda^{1/d}(t)$. It would be interesting to give a rigorous interpretation of the above heuristic argument.)

Below, we assume that the tail $1 - G(u)$ of the distribution function G is continuous, strictly monotone and strictly positive for all sufficiently large u. Its inverse $(1 - G)^{-1}(.)$ is well-defined, continuous and strictly monotone on $(0, \delta)$, for some $\delta \in (0, 1)$, and $(1 - G)^{-1}(0+) = +\infty$. Put

$$H_{1,T}(u) := T(1 - G)(A(T) + uB(T)), \tag{6.8}$$

$$H_{2,T}(u) := T(1 - G)\Big(\sqrt{A^2(T) + u^2B^2(T)} + uB(T)\Big), \tag{6.9}$$

$T \ge 1$, where $A(T), B(T) > 0$ are normalizing constants to be specified below. Also, recall, that a real-valued function $L(t), t > 0$, is said to be *regularly varying with exponent* $\theta \in \mathbf{R}$ if, for any $a > 0$, the ratio $L(at)/L(t) \to a^\theta$ as $t \to \infty$ (see, e.g., Bingham, Goldie, Teugels (1987)).

Theorem 6.1. *Assume that there exist* $A(T) = (1-G)^{-1}(1/T)$ *and a regularly varying at infinity function* $B(T) > 0,\ T \geq 1$, *with exponent* $\theta \in [0, 1/d)$, *such that for any* $u \in \mathbf{R}$ *there exist limits*

$$\lim_{T\to\infty} H_{1,T}(u) \equiv H(u) \in [0, +\infty], \tag{6.10}$$

and

$$\lim_{T\to\infty} \int_{\mathbf{R}^d} H_{2,T}(|x|)\, dx \equiv h(\theta) \in (0, \infty). \tag{6.11}$$

Then, $\lambda(t)$, *defined in (6.7), regularly varies as* $t \to \infty$ *with exponent* $-d/(1-\theta d)$; *i.e., there exists a slowly varying function* $L(t)$ *such that*

$$\lambda(t) = L(t) t^{-d/(1-\theta d)}. \tag{6.12}$$

Remark 6.1. Condition (6.10) implies that G is asymptotically max-stable, see e.g. Leadbetter, Lindgren, Rootzen (1983), Bingham, Goldie, Teugels (1987). Namely, for any $u \in \mathbf{R}$,

$$\lim_{n\to\infty} G^n(A(n) + uB(n)) = e^{-H(u)}.$$

The limit function has one of the three well-known parametric forms (type I, II or III extreme value distributions). Under the assumptions of Theorem 6.1, there are only two possibilities: either

$$H(u) = e^{-cu}, \tag{6.13}$$

$u \in \mathbf{R}$, with $c > 0$ (type I distribution), or

$$H(u) = (1 + cu)^{-\gamma}, \tag{6.14}$$

if $u > -1/c,\ H(u) = +\infty$ if $u \leq -1/c$ (type II distribution), where $c, \gamma > 0$ are parameters ($\gamma > d$, according to condition (6.11)). In the latter case, $1 - G(u)$ is necessarily regularly varying with exponent $-\gamma$ (see, e.g., Leadbetter, Lindgren, Rootzen (1983)). In particular,

$$1 - G(u) \sim c_1 u^{-\gamma} \qquad (u \to \infty, c_1 > 0)$$

satisfies conditions of Theorem 6.1 with

$$A(T) = B(T) = (c_1 T)^{1/\gamma}$$

$$H(u) = (1 + u)^{-\gamma}, \qquad u > -1,$$

$$h(\theta) = h(1/\gamma) = \int_{\mathbf{R}^d} (|x| + \sqrt{|x|^2+1})^{-\gamma} dx,$$

yielding

$$\lambda(t) \sim c_2 t^{-d\gamma/(\gamma-d)}, \tag{6.15}$$

with

$$c_2 = \lambda^{-d/(\gamma-d)} \Gamma\left(\frac{2\gamma-d}{\gamma-d}\right) 2^{-d\gamma/2(\gamma-d)}\, h(1/\gamma)^{-\gamma/(\gamma-d)}\, c_1^{-d/(\gamma-d)},$$

see the proof of Theorem 6.1.

The class of probability distributions attracted to a type I distribution (6.13) contains many familiar distributions such as normal, exponential, fractional (stretched) exponential (Weibull), etc. For example,

$$1 - G(u) \sim \exp[-c_3 u^\alpha], \qquad (u \to \infty, \alpha, c_3 > 0)$$

satisfies Theorem 6.1 with

$$\theta = 0, \qquad A(T) = c_3^{-1/\alpha} (\log T)^{1/\alpha}, \qquad B(T) = \alpha^{-1} c_3^{-1/\alpha} (\log T)^{1/\alpha - 1},$$

$$H(u) = e^{-u}, \qquad h(0) = \int_{\mathbf{R}^d} e^{-|x|} dx = 2\pi^{d/2} \Gamma(d)/\Gamma(d/2),$$

so that

$$\lambda(t) \sim c_4 t^{-d} (\log t)^{d(\alpha-1)/\alpha}, \tag{6.16}$$

with

$$c_4 = d^{d(\alpha-1)/\alpha}\, \alpha^d\, 2^{-1-d/2}\, \pi^{-d/2} c_3^{d/\alpha}\, \Gamma(d/2)/\Gamma(d).$$

By strengthening slightly the assumptions of Theorem 6.1 one can show that the process $\{(x_j^*(t), g_j^*(t))\}$ itself converges in distribution, after an appropriate scaling, to a limit process, giving rise to a limit quasi-Voronoi tessellation.

Let $\mathcal{N}(X)$ be the set of all locally finite point measures on an open set $X \subset \mathbf{R}^n$, $n \geq 1$, with the topology of vague convergence of measures (see, e.g. Kallenberg (1986)). Write $\Rightarrow$ for the convergence in distribution of random elements in $\mathcal{N}(X)$ (the weak convergence of point processes).

Theorem 6.2. *Assume, in addition to conditions imposed in Theorem 6.1, that there exists the limit*

$$\lim_{T\to\infty} \frac{A(T)}{B(T)} \equiv R \in [0, +\infty]. \tag{6.17}$$

Then, one can find normalizing constants $a_T, b_T \to \infty$ $(T \to \infty)$, $b_T^d \sim const/\lambda(T)$, such that the rescaled process

$$\big\{(x_j^*(tT)/b_T, (g_j^*(tT) - a_T)T/b_T)\big\}, \qquad t > 0, \tag{6.18}$$

converges, as $T \to \infty$, in the sense of weak convergence of finite-dimensional distributions, to an $\mathcal{N}(\mathbf{R}^d \times \mathbf{R})$-valued process

$$\{(x_{j,\infty}^*(t), g_{j,\infty}^*(t))\}, \qquad t > 0. \tag{6.19}$$

For each $t > 0$, the limit process (6.19) can be identified with the set of vertices of the cone envelope

$$S_\infty(t,x) = \begin{cases} \sup_j (2tg_{j,\infty} - \sqrt{2}|x - x_{j,\infty}|), & \textit{if } R = \infty; \\ \sup_j (t(g_{j,\infty} + R)^2 - \sqrt{2}(g_{j,\infty} + R)|x - x_{j,\infty}|), & \textit{if } R < \infty, \end{cases} \tag{6.20}$$

where $\{(x_{j,\infty}, g_{j,\infty})\}$ is a Poisson process on $\mathbf{R}^d \times (H_-, \infty)$, $H_- =: \inf\{u : H(u) < +\infty\}$, having intensity measure $-\lambda\, dx\, dH(u)$.

Of course, the mean density $\lambda(t)$ (6.5) of cells is the simplest statistical parameter of the quasi-Voronoi tessellation $\{C_j(t)\}$, with many others (e.g. the distributions of the volume, surface area, length of edges, etc., of a typical cell, and the corresponding averages) are of interest. However, exact analytic formulas are often difficult to obtain even for the classical Voronoi tessellation, usually being replaced by Monte-Carlo simulations, (see, e.g. Møller (1994), van de Weygaert (1991), and Janicki, Surgailis, Woyczynski (1995), where the correlation dimension and geometric thermodynamic temperature were estimated for the associated passive tracer flows). Sahni, Sathyaprakash, Shandarin (1994) obtained numerical histograms of the distribution of the void (cell) diameters for different times, in the adhesion model of the large scale structure of the Universe without forcing.

The distribution of the cell diameter in our model of forced Burgers' turbulence can be characterized in terms of what we call the *Palm cell function* which, by definition is the conditional probability

$$p(r,t) = P\Big(\{|y - x_j| \le r\} \subset C_j(t) \;\Big|\; x_j = x \in \{x_j^*(t)\}\Big) \tag{6.21}$$

that a cell $C_j(t)$ contains a ball of radius $r > 0$ centered at x_j, under the condition that the point $x_j = x$ is fixed. By stationarity, (6.21) does not depend on x which we can always assume to be 0. To evaluate $p(r,t)$ analytically, note that the condition

$$\{|y - x_j| \le r\} \subset C_j(t)$$

is equivalent to the condition

$$tg_k^2 - \sqrt{2}(|x_k - x| - r)g_k < tg_j^2 - \sqrt{r}g_j$$

for all $k \neq j$. Then, as in (6.6-7), we obtain

$$p(r,t) = \frac{\lambda(r,t)}{\lambda(0,t)}, \tag{6.22}$$

where $\lambda(0,t) = \lambda(t)$ and

$$\begin{aligned}\lambda(r,t) =& P\Big(2tg_k < \sqrt{2}(|x_k - x| - r) \\ &+ \sqrt{2(|x_k - x| - r)^2 + 4tg_j(tg_j - \sqrt{2}r)}, \text{ for all } k \neq j, x_j \in dx\Big)/dx \\ =& \lambda \int_{\sqrt{2}r/t}^{\infty} \exp\Big[-\lambda(\sqrt{2}t)^d \int_{\mathbf{R}^d} (1-G)\Big(|x| - r/\sqrt{2}t \\ &+ \sqrt{(|x| - r/\sqrt{2}t)^2 + u(u - \sqrt{2}r/t)}\Big) dx\Big] dG(u).\end{aligned} \tag{6.23}$$

Theorem 6.3. *Under the conditions and notation of Theorem 6.2, for any $r, t > 0$, there exists the limit*

$$\lim_{T\to\infty} p(rb_T, tT) = p_\infty(r,t), \tag{6.24}$$

which coincides with the Palm cell function for the scaling limit quasi-Voronoi tessellation function generated by $S_\infty(t,x)$ (6.20). In particular, in the case $R = +\infty$ and $H(u) = e^{-u}$, one has

$$p_\infty(r,t) = e^{-\sqrt{2}r/t}. \tag{6.25}$$

7. Proofs of Theorems 6.1-3.

Proof of Theorem 6.1. By (6.7),

$$\lambda(t) = \lambda \int_0^\delta \exp\big[-\lambda(\sqrt{2}t)^d h(\theta) K(v)\big]\, dv + O(\exp[-ct^d]),$$

where $\delta, c > 0$, and

$$K(v) = \int_{\mathbf{R}^d} (1-G)\Big(|x| + \sqrt{|x|^2 + ((1-G)^{-1}(v))^2}\Big)\, dx/h(\theta).$$

In fact, $\lambda(t) = \lambda\int_0^\delta \exp[\ldots]\,dv + \lambda_1(t)$, where $\lambda_1(t) = \lambda\int_0^{u_\delta} e^{-c(u)t^d} dG(u)$, $u_\delta = (1-G)^{-1}(\delta) < \infty$, and $c(u) = \lambda 2^{d/2}\int_{\mathbf{R}^d}(1-G)(|x| + \sqrt{|x|^2+u^2})\,dx \geq c = c_\delta > 0$ for $0 \leq u \leq u_\delta$, which implies that $\lambda_1(t) = O(e^{-ct^d})$. Hence, it suffices to show that

$$K(v) \sim vB^d(1/v) \qquad (v \to 0). \tag{7.1}$$

Indeed, the right-hand side of (7.1), and hence $K(v)$, vary regularly with exponent $1-\theta d > 0$ as $v \to 0$. Its generalized inverse

$$K^{(-1)}(z) = \inf\{v \in (0,\delta] : K(v) > z\} \tag{7.2}$$

is monotonically increasing on $(0, K(\delta))$, satisfies asymptotic relation

$$K(K^{(-1)}(z)) \sim z \qquad (z \to 0), \tag{7.3}$$

and varies regularly with exponent $1/(1-\theta d)$, i.e.,

$$K^{(-1)}(z) = L_1(1/z)\, z^{1/(1-\theta d)}, \tag{7.4}$$

where $L_1(.)$ is a slowly varying function (see, e.g., Bingham, Goldie, Teugels (1987)). By Karamata's Tauberian Theorem (see, e.g., Feller (1971))

$$\begin{aligned}
\lambda(t) &\sim \lambda\int_0^\delta \exp[-\lambda(\sqrt{2}t)^d\, h(\theta)\, K(v)]\,dv \\
&\sim \lambda\int_0^{K(\delta)} \exp[-\lambda(\sqrt{2}t)^d\, h(\theta) z]\,dK^{(-1)}(z) \\
&\sim \lambda\Gamma\left(\frac{2-\theta d}{1-\theta d}\right) K^{(-1)}\left(1/\lambda(\sqrt{2}t)^d h(\theta)\right) \\
&\sim cL_1(t^d)\, t^{-d/(1-\theta d)},
\end{aligned}$$

where

$$c = \lambda^{-\theta d/(1-\theta d)}\Gamma\left(\frac{2-\theta d}{1-\theta d}\right) 2^{-d/2(1-\theta d)}\, h(\theta)^{-1/(1-\theta d)}.$$

This proves (6.12). Finally, (7.1) immediately follows from (6.11) and the definition of $A(T)$; indeed,

$$K(v) = vB^d(1/v)\int_{\mathbf{R}^d} H_{2,1/v}(|x|)\,dx/h(\theta) \sim vB^d(1/v).$$

Theorem 6.1 is proved.

Proof of Theorem 6.2. Put

$$b_T^{-d} = K^{(-1)}(T^{-d}), \tag{7.5}$$

$$a_T = A(b_T^d), \tag{7.6}$$

where $K^{(-1)}(.)$ and $A(.)$ are the same as in Theorem 6.1. Consider the rescaled Poisson process

$$x_{j,T} = \frac{x_j}{b_T}, \qquad g_{j,T} = \frac{(g_j - a_T)T}{b_T}, \tag{7.7}$$

see (6.18), with intensity measure $-\lambda\, dx\, d\tilde{H}_{1,T}(u)$, where

$$\tilde{H}_{1,T}(u) = b_T^d(1-G)(a_T + ub_T/T). \tag{7.8}$$

Then, for each $u \in (H_-, \infty)$,

$$\lim_{T\to\infty} \tilde{H}_{1,T}(u) = H(u). \tag{7.9}$$

Indeed, according to (7.5-6),

$$\tilde{H}_{1,T}(u) = b_T^d(1-G)(A(b_T^d) + uB(b_T^d)\theta_T) = H_{1,b_T^d}(u\theta_T), \tag{7.10}$$

where $H_{1,T}(.)$ was defined in (6.8), and

$$\theta_T = b_T/(TB(b_T^d)) \to 1 \qquad (T \to \infty), \tag{7.11}$$

according to (7.1-2) and (7.5). The convergence (6.10) being uniform in any compact interval of $(H_-, +\infty)$ (see Albeverio, Molchanov, Surgailis (1994), proof of Theorem 9), from (6.10) and (7.10-11) we obtain (7.9). This guarantees the convergence in $\mathcal{N}(\mathbf{R}^d \times (H_-, \infty))$ of the corresponding Poisson process:

$$\{(\tilde{x}_{j,T}, \tilde{g}_{j,T})\} \Longrightarrow \{(x_{j,\infty}, g_{j,\infty})\}, \tag{7.12}$$

where

$$\{(\tilde{x}_{j,T}, \tilde{g}_{j,T})\} = \{(x_{j,T}, g_{j,T}) : g_{j,T} > H_-\}.$$

Below, for the sake of simplicity, we consider only the case $H_- = -\infty$. Then,

$$\{(\tilde{x}_{j,T}, \tilde{g}_{j,T})\} = \{(x_{j,T}, g_{j,T})\}.$$

Write $\{(x^*_{j,T}(t), g^*_{j,T}(t))\}$ for the point process of (6.18). Let $f \in C_0(\mathbf{R}^d \times \mathbf{R})$. Then, according to (6.4) and (7.5-6),

$$\begin{aligned} &\sum f(x^*_{j,T}(t), g^*_{j,T}(t)) \\ =&\sum f(x_{j,T}, g_{j,T})\mathbf{1}\Big(2tg_{k,T} < \sqrt{2}|x_{j,T} - x_{k,T}| \\ &+ \sqrt{2|x_{j,T} - x_{k,T}|^2 + 4t^2(g_{j,T} + R_T)^2} - 2tR_T, \text{ for all } k \neq j\Big) \\ =:& \Theta_{t,T}(\{(x_{j,T}, g_{j,T})\}), \end{aligned} \tag{7.13}$$

where

$$R_T = \frac{a_T T}{b_T} \sim \frac{a_T}{B(b_T^d)} = \frac{A(b_T^d)}{B(b_T^d)} \to R \qquad (T \to \infty), \tag{7.14}$$

see (7.11) and (6.17). Hence, for almost all realizations $\{(x_j, g_j)\} \in \mathcal{N}(\mathbf{R}^d \times \mathbf{R})$ of the Poisson process $\{(x_{j,\infty}, g_{j,\infty})\}$ we easily obtain

$$\Theta_{t,T}(\{(x_j, g_j)\}) \to \Theta_{t,\infty}(\{(x_j, g_j)\}), \tag{7.15}$$

where

$$\begin{aligned} \Theta_{t,\infty}(\{(x_j, g_j)\}) = &\sum f(x_j, g_j)\mathbf{1}\Big(2t(g_k + R) < \sqrt{2}|x_j - x_k| \\ &+\sqrt{2|x_j - x_k|^2 + 4t^2(g_j + R)^2}, \text{ for all } k \neq j\Big) \end{aligned} \tag{7.16}$$

if $R < \infty$, while

$$\Theta_{t,\infty}(\{(x_j, g_j)\}) = \sum f(x_j, g_j)\mathbf{1}\Big(2tg_k < \sqrt{2}|x_j - x_k| + 2tg_j, \text{ for all } k \neq j\Big), \tag{7.17}$$

if $R = +\infty$. The limit functional (7.16-17) is continuous on $\mathcal{N}(\mathbf{R}^d \times \mathbf{R})$ with the exception of a set having (Poisson) probability zero. Hence, the desired convergence follows from (7.12) and a well-known result about weak convergence (see Kallenberg (1986), 15.4.2). Finally, the identification of the limit process given in the theorem follows from the definition (7.16-17) of $\Theta_{t,\infty}$. This concludes the proof of Theorem 6.2.

Proof of Theorem 6.3. According to (6.22-23)

$$p(rb_T, tT) = \lambda_T(r,t)/\lambda_T(0,t),$$

where $\lambda_T(r,t)$ is the corresponding intensity for the rescaled process (6.18), i.e.,

$$\begin{aligned} \lambda_T(r,t) =&P\Big(2t(g_{k,T} + R_T) < \sqrt{2}(|x_{k,T} - x| - r) \\ &+ \sqrt{2(|x_{k,T} - x| - r)^2 + 4t^2(g_{j,T} + R_T)(g_{j,T} + R_T - \sqrt{2}r/t)}, \\ &\qquad \text{for all } k \neq j; x_{j,T} \in dx\Big)/dx \\ =& -\lambda \int_{-R_T + \sqrt{2}r/t} \exp\Big[-\lambda \int_{\mathbf{R}^d} \tilde{H}_{1,T}\Big(\frac{|x| - r}{\sqrt{2}t} \\ &+ \sqrt{(|x| - r)^2/2t + (u + R_T)(u + R_T - \sqrt{2}r/t)} - R_T\Big)dx\Big] d\tilde{H}_{1,T}(u), \end{aligned}$$

(see (7.13, 6.23)) with $\tilde{H}_{1,T}(u)$ given by (7.10) and $\tilde{H}_{1,T}(u) \to H(u)$, see (7.9). As in the proofs of Theorems 6.1-2, by considering the cases $\lim_{T\to\infty} R_T \equiv R < \infty$ and $R = \infty$, separately, one can prove the existence of the limit $\lim_{T\to\infty} \lambda_T(r,t) \equiv \lambda_\infty(r,t) > 0$, and hence also (6.24). In the case $R = \infty$ and $H(u) = e^{-u}$ this limit can be explicitly evaluated. Indeed,

$$\begin{aligned}\lambda_\infty(r,t) =& -\lambda \int_{-\infty}^{\infty} \exp\Big[-\lambda \int_{\mathbf{R}^d} H\Big(\frac{|x|-2r}{\sqrt{2}t} + u\Big)\, dx\Big]\, dH(u) \\ =&\lambda \int_{-\infty}^{\infty} \exp\Big[-e^{-u+\sqrt{2}r/t} \int_{\mathbf{R}^d} e^{-|x|/\sqrt{2}t} dx\Big] e^{-u} du \\ =&\lambda e^{-\sqrt{2}r/t} \Big(\int_{\mathbf{R}^d} e^{-|x|/\sqrt{2}t} dx\Big)^{-1} = e^{-\sqrt{2}r/t} \lambda_\infty(0,t),\end{aligned}$$

yielding (6.25).

8. Nonhomogeneous initial data.

The "geometric" solution of the variational problem (3.3) in Theorem 4.1 can be extended to a nonzero "discrete" initial potential $S_0(x)$ of the form

$$S_0(x) = \sum_j \xi_j \mathbf{1}[x = y_j], \tag{8.1}$$

where $\xi_j > 0$ and $y_j \in \mathbf{R}^d$ are isolated points. Namely, under the growth conditions (3.9-10), $S(t,x)$ coincides with the upper envelope

$$S(t,x) = \sup_j c_j(t,x) \vee p_j(t,x) \tag{8.2}$$

of cones $c_j(t,x)$ (4.4a) and paraboloids

$$p_j(t,x) = (\xi_j - |x - y_j|^2/2t) \vee 0. \tag{8.3}$$

Then, the corresponding inviscid solution $\vec{v}(t,x) = -\nabla S(t,x)$ can be approached by a smooth approximation $\Phi_n(.)$, $\xi_n(.)$, as in Theorem 4.2.

In the unforced case $\Phi(.) \equiv 0$, (8.2) yields the well-known formula

$$\vec{v}(t,x) = \begin{cases} (x - y_{j^*})/t, & \text{if } \xi_{j^*} > |x - y_{j^*}|^2/2t; \\ 0, & \text{otherwise,} \end{cases} \tag{8.4}$$

where (y_{j^*}, ξ_{j^*}) satisfies

$$\xi_{j^*} - \frac{|x - y_{j^*}|^2}{2t} = \sup_j \left(\xi_j - \frac{|x - y_j|^2}{2t}\right), \tag{8.5}$$

see, e.g., Albeverio, Molchanov, Surgailis (1994), Molchanov, Surgailis, Woyczynski (1995). The corresponding quasi-Voronoi tessellation consists of (connected) cells

$$D_j(t) = \big\{x \in \mathbf{R}^d; S(t,x) = p_j(t,x)\big\}.$$

Statistical properties of the point process $\{(y_j^*(t), \xi_j^*(t))\}$ of apexes of paraboloids $p_{j^*}(t,x)$, centers of our quasi-Voronoi cells $D_j(t)$, were discussed in Albeverio, Molchanov, Surgailis (1994), under the Poisson hypothesis of the initial process $\{(y_j, \xi_j)\}$, and similar conditions on the p.d.f. $Q(u) := P(\xi_j \le u)$. Let

$$\nu(t) = P(y_j^*(t) \in dx)/dx$$

be the corresponding density; $\nu = \nu(0) = P(y_j \in dx)/dx$. Then, as in (6.7),

$$\nu(t) = \nu \int_0^\infty \exp\Big[-\nu(\sqrt{2t})^d \int_{R^d} (1-Q)(|x|^2+u)\,dx\Big]\,dQ(u), \tag{8.6}$$

which suggests a much slower decay compared with $\lambda(t)$; roughly

$$\nu(t) \approx O(\sqrt{\lambda(t)}) \qquad (t \ \to \infty). \tag{8.7}$$

In other words, *typical cells in the unforced Burgers' turbulence are much smaller, roughly the square root of the size of cells in the forced turbulence*, indicating that in the latter case the formation of the cell structure occurs much faster.

Relation (8.7) can be rigorously established under additional conditions on the p.d.f. $Q(u)$ and $F(u) = P(h_j \le u)$. For example, for

$$1 - Q(u) \sim 1 - F(u) \sim c_1 u^{-\gamma/2} \qquad (\gamma > d, u \to \infty),$$

we get that

$$\lambda(t) \sim c_2 t^{-d\gamma/(\gamma-d)},$$

whereas

$$\nu(t) \sim c_3 t^{-d\gamma/2(\gamma-d)},$$

where the constants $c_2, c_3 > 0$ can be explicitly found; see Remark 6.1. For exponentially decaying tails

$$1 - Q(u) \sim 1 - F(u) \sim \exp[-c_4 u^{\alpha/2}] \qquad (c_4, \alpha > 0, u \to \infty),$$

one obtains that

$$\lambda(t) \sim c_5 t^{-d} (\log t)^{d(\alpha-1)/\alpha},$$

whereas

$$\nu(t) \sim c_6 t^{-d/2} (\log t)^{d(\alpha-1)/\alpha},$$

which again confirms the hypothesis (8.7), up to a slowly varying factor.

Of course, the present paper is only the first attempt to rigorously discuss the formation and evolution of the cellular structure in forced Burgers turbulence; our model (4.1) being rather a " caricature" of a more realistic potential (e.g., Gaussian). However, a discussion of such potentials may require more advanced techniques, in particular, the methods of localization theory and spectral analysis for Schrödinger operators (see, e.g. Molchanov (1994) , Molchanov, Surgailis, Woyczynski (1995)).

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